Roots of the Cylindrical Shell Characteristic Equation for Harmonic Circumferential Edge Loading

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The roots of Flügge's complete characteristic equation for small deformations of circular cylindrical shells loaded along the curved edges are compared with those of various widely used approximations to the equation. The roots are found to be extremely sensitive to small changes in the coefficients of the equations when the half-wave length of the circumferential variation of the applied edge load is of the order of three wall thicknesses or less. For these rapid circumferential load variations the roots of the complete equation not only deviate widely from those of the approximate equations but cease to be all complex as has been commonly supposed.

Nomenclature

h = cylinder wall thickness
k = thickness-radius ratio parameter $[k = (1/12)(h^2/R^2)]$
n = number of circumferential waves
$p_{n1}, q_{n1}, p_{n2}, q_{n2}$ = nomenclature for roots of characteristic equation
R = cylinder radius
u,v,w = middle surface longitudinal, circumferential, and radial displacements, respectively
x = distance along cylinder generator
α_1–α_9 = quantities in coefficients of characteristic equation
θ = circumferential angular location of point on middle surface
λ = characteristic number
ν = Poisson's ratio of shell material
ξ = nondimensional longitudinal distance ($\xi = x/R$)

Introduction

IN the analysis of circular cylindrical shells, the theory developed by Flügge[1] is used as a standard for comparison with more approximate theories. For cylindrical shells under harmonically varying edge loading, it is customary to compare the roots of the approximate characteristic equation with those of Flügge's characteristic equation. In doing so, however, the basis of comparison has not been the complete characteristic equation but a simplified equation, also introduced by Flügge, which differs from the complete equation by the deletion of terms of the order of the square of the thickness-radius ratio compared to unity.

Because of the small differences between the actual and simplified coefficients of the characteristic equation it has been supposed that the roots of the one were almost identical with those of the other. The present investigation, motivated by some unexpected results obtained in conjunction with the work leading to Ref. 2, shows, however, that the supposition is not correct when the half-wave length of the circumferential variation of the applied edge load is of the order of three wall thickness or less. The results of the investigation also challenge several other common suppositions: 1) that the roots of Flügge's characteristic equation are all complex and 2) that the roots of Morley's equation[3] are always a good approximation to those of Flügge's simplified characteristic equation. The impact of these results is mitigated, however, by the fact that they occur only for such rapid circumferential load variations that the validity of the applicability of the Love-Kirchhoff shell theory to isotropic shells is questionable. The question of the actual longitudinal variation of exponentially decaying solutions of the equations of three-dimensional isotropic elasticity theory is being investigated.

The results reported herein were obtained by means of a numerical investigation of the roots of the various equations using a digital computer. The computer subroutine used (Aerospace Corporation Subroutine ASC MULE) is based upon a modification of Muller's method.[4] The author is indebted to J. F. Holt for having developed the subroutine which was used and to A. Linger for coding and guiding the problem in its numerical stages.

Characteristic Equations for Circular Cylindrical Shells Loaded Along the Curved Edges

Flügge's equations for small deformations of edge-loaded circular cylindrical shells can be expressed as[1]

$$\left[\frac{\partial^2}{\partial\xi^2} + \frac{1-\nu}{2}(1+k)\frac{\partial^2}{\partial\theta^2}\right]u + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial\xi\partial\theta} + \left[\nu - k\left(\frac{\partial^2}{\partial\xi^2} - \frac{1-\nu}{2}\frac{\partial^2}{\partial\theta^2}\right)\right]\frac{\partial w}{\partial\xi} = 0 \quad \text{(1a)}$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial\xi\partial\theta} + \left[\frac{1-\nu}{2}(1+3k)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right]v + \left(1 - \frac{3-\nu}{2}k\frac{\partial^2}{\partial\xi^2}\right)\frac{\partial w}{\partial\theta} = 0 \quad \text{(1b)}$$

$$\left[\nu - k\left(\frac{\partial^2}{\partial\xi^2} - \frac{1-\nu}{2}\frac{\partial^2}{\partial\theta^2}\right)\right]\frac{\partial u}{\partial\xi} + \left(1 - \frac{3-\nu}{2}k\frac{\partial^2}{\partial\xi^2}\right)\frac{\partial v}{\partial\theta} + \left[1 + k\left(\frac{\partial^4}{\partial\xi^4} + 2\frac{\partial^4}{\partial\xi^2\partial\theta^2} + \frac{\partial^4}{\partial\theta^4} + 2\frac{\partial^2}{\partial\theta^2} + 1\right)\right]w = 0 \quad \text{(1c)}$$

Since Eqs. (1) are linear and the operators are commutative, a single equation for, say, w can be obtained by simple manipulations as

$$\left[\alpha_1\frac{\partial^8}{\partial\xi^8} + \alpha_2\frac{\partial^8}{\partial\xi^6\partial\theta^2} + \alpha_3\frac{\partial^8}{\partial\xi^4\partial\theta^4} + \alpha_4\frac{\partial^4}{\partial\xi^2\partial\theta^2}\left(\frac{\partial^4}{\partial\theta^4} - 1\right) + \alpha_5\frac{\partial^4}{\partial\theta^4}\left(\frac{\partial^2}{\partial\theta^2} + 1\right)^2 + \alpha_6\frac{\partial^6}{\partial\xi^6} + \alpha_7\frac{\partial^6}{\partial\xi^4\partial\theta^2} + \alpha_8\frac{\partial^4}{\partial\xi^2\partial\theta^2}\left(\frac{\partial^2}{\partial\theta^2} + 1\right) + \alpha_9\frac{\partial^4}{\partial\xi^4}\right]w = 0 \quad (2)$$

where

$$\alpha_1 = (1-k)(1+3k) \quad \text{(3a)}$$

$$\alpha_2 = 4 + (11 - 3\nu)k + 9[(1-\nu)/2]k^2 \quad \text{(3b)}$$

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$$\alpha_3 = 6 + (4 - 3\nu)k - \nu^2k^2 \tag{3c}$$

$$\alpha_4 = 4 + [(7 - 3\nu)/2]k + 3[(1 - \nu)/2]k^2 \tag{3d}$$

$$\alpha_5 = 1 + k \tag{3e}$$

$$\alpha_6 = 2\nu(1 + 3k) \tag{3f}$$

$$\alpha_7 = 6 + 3(2 - \nu + \nu^2)k \tag{3g}$$

$$\alpha_8 = 2(4 - \nu) + (7 - 5\nu)k + 3(1 - \nu)k^2 \tag{3h}$$

$$\alpha_9 = (1 + 3k)\ [(1 - \nu^2)/k + 1] \tag{3i}$$

For edge-loaded cylindrical shells we consider solutions of Eq. (2) of the form

$$w = e^{\lambda_n\xi} \cos n\theta \tag{4}$$

Then λ_n is determined by the following characteristic equation obtained by substituting Eq. (4) into Eq. (2):

$$\alpha_1\lambda_n^8 + (\alpha_6 - \alpha_2n^2)\lambda_n^6 + (\alpha_9 - \alpha_7n^2 + \alpha_3n^4)\lambda_n^4 + [\alpha_8 - \alpha_4(n^2 + 1)]n^2(n^2 - 1)\lambda_n^2 + \alpha_5n^4(n^2 - 1)^2 = 0 \tag{5}$$

It is customary to make approximations in the characteristic equation in order to obtain an equation whose roots are expressible in a simpler form or to delete terms which are small compared to unity and therefore of apparent negligible importance. Thus, we have Flügge's simplified characteristic equation[2] obtained by deleting terms in the coefficients which are of the order of k compared to unity, i.e., Eq. (5) with

$$\alpha_1 = \alpha_5 = 1 \tag{6a}$$

$$\alpha_2 = \alpha_4 = 4 \tag{6b}$$

$$\alpha_3 = \alpha_7 = 6 \tag{6c}$$

$$\alpha_6 = 2\nu \tag{6d}$$

$$\alpha_8 = 2(4 - \nu) \tag{6e}$$

$$\alpha_9 = (1 - \nu^2)/k \tag{6f}$$

A similar approximation is given by Morley[3] in which

$$\alpha_1 = \alpha_5 = 1 \tag{7a}$$

$$\alpha_2 = \alpha_4 = 4 \tag{7b}$$

$$\alpha_3 = \alpha_7 = 6 \tag{7c}$$

$$\alpha_6 = 2 \tag{7d}$$

$$\alpha_8 = 6 \tag{7e}$$

$$\alpha_9 = (1 - \nu^2)/k + 1 \tag{7f}$$

Another approximation due to Donnell[5] is not discussed here since it is similar to that given by Morley and is known to yield the same results when n^2 is large compared to unity.

Results and Discussion

The reduction of the coefficients from those given by Eq. (3) to those given by Eqs. (6) is apparently a trivial one in that small terms of the order of $(h/R)^2$ compared to unity are neglected. The further change from Eqs. (6) to Eqs. (7) is less trivial. The effect of the change however, would be expected to be insignificant for large n since the change in the coefficients is of the order of $1/n^2$ compared to unity. A numerical comparison of the roots of the three equations defined by Eq. (5) together with Eqs. (3, 6, or 7) indicates, however, that the biquartic equation is ill-conditioned so that small changes in the coefficients lead to large changes in the roots, the differences increasing as n, the number of circumferential waves, increases.

The roots of the three equations were obtained with the use of a digital computer and a computer subroutine for polynominal root extraction based on Muller's method.[4] The value of n was varied from 0 to 100 by steps of unity. Poisson's ratio ν was taken equal to 0.3 in all calculations. Finally R/h was allowed to take the values 10, 20, 50, 100, 200, 500, 1000. The roots are expressed as

a) **Values of q_{n1}**

b) **Values of q_{n2}**

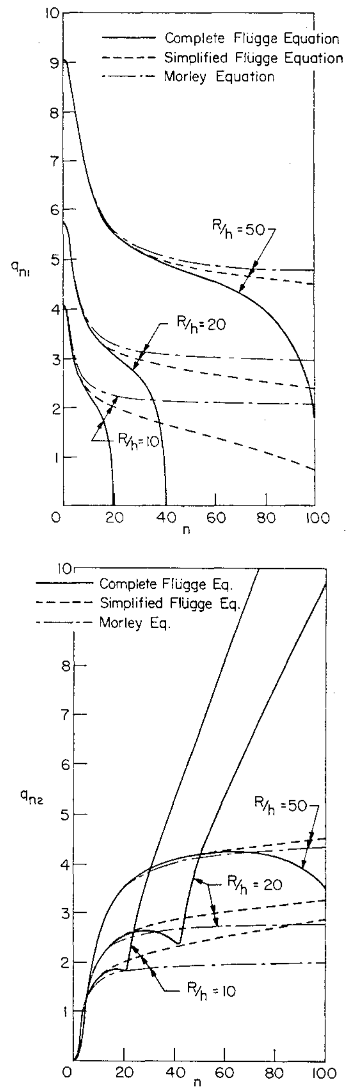

Fig. 1 Comparison of imaginary parts of roots of various characteristic equations.

$$(\lambda_n)_{1,2,3,4} = \pm\ p_{n1} \pm\ iq_{n1} \tag{8a}$$

$$(\lambda_n)_{5,6,7,8} = \pm\ p_{n2} \pm\ iq_{n2} \tag{8b}$$

if all of the roots are complex, or as

$$(\lambda_n)_{1,2} = \pm\ p_{n1} \tag{9a}$$

$$(\lambda_n)_{3,4} = \pm\ q_{n1} \tag{9b}$$

$$(\lambda_n)_{5,6,7,8} = \pm\ p_{n2} \pm\ i\ q_{n2} \tag{9c}$$

when four roots are real and the remaining four are complex. Some values of p_{n1}, q_{n1}, p_{n2}, and q_{n2} are given in Table 1 for R/h equal to 10.† Significant differences in the roots were observed in the range of n considered for $R/h \leq 50$.

The calculated results indicate the following trends. The roots of the complete characteristic equation of Flügge's theory of shells are all complex until n reaches a value of about $2R/h$. For larger values of n four roots are real and four are complex.

All roots of the simplified characteristic equation of Flügge's theory are in good agreement with those of the exact equation

† Detailed tables are available for $R/h = 10$, 20, and 50, $n =$ 0 to 100 by steps of unity. These may be obtained from the author upon request.

Table 1 Values of the roots of the various characteristic equations for edge-loaded cylindrical shells

	Complete Flügge equation				Simplified Flügge equation				Morley equation			
n	p_{n1}	q_{n1}	p_{n2}	q_{n2}	p_{n1}	q_{n1}	p_{n2}	q_{n2}	p_{n1}	q_{n1}	p_{n2}	q_{n2}
0	4.04809	4.08501	...	...	4.04632	4.08322	...	...	4.00378	4.12678	...	...
2	4.56907	3.63829	0.45591	0.37917	4.56742	3.63670	0.45623	0.37934	4.52933	3.68597	0.46010	0.37443
4	6.13106	2.98083	2.02127	1.03747	6.13019	2.98198	2.02206	1.03717	6.09840	3.04794	2.03260	1.01588
6	8.05766	2.62495	3.95179	1.38593	8.05902	2.63434	3.95090	1.38482	8.03034	2.72090	3.96509	1.34348
8	10.03733	2.41426	5.94360	1.57791	10.04520	2.44164	5.93651	1.57693	10.01793	2.54956	5.95287	1.51501
10	12.02013	2.25486	7.95461	1.69922	12.04268	2.31654	7.93309	1.70111	12.01600	2.44608	7.95103	1.61858
12	13.99222	2.10365	9.98198	1.78061	14.04310	2.22571	9.93237	1.79075	14.01655	2.37703	9.95163	1.68767
14	15.94332	1.92978	12.03192	1.83319	16.04442	2.15449	11.93230	1.86054	16.01769	2.32773	11.95279	1.73701
16	17.85878	1.69363	14.11788	1.85942	18.04604	2.09549	13.93232	1.91786	18.01888	2.29076	13.95401	1.77399
18	19.70916	1.30157	16.26891	1.85739	20.04777	2.04459	15.93222	1.96685	20.02000	2.26202	15.95514	1.80274
20[a]	21.88518	20.93239	18.57057	1.83790	22.04954	1.99932	17.93196	2.00996	22.02099	2.23904	17.95613	1.82574
22[a]	24.88615	20.83835	21.12724	2.13681	24.05133	1.95812	19.93151	2.04878	24.02186	2.22024	19.95701	1.84454
24[a]	27.39666	21.95880	23.30374	2.69095	26.05316	1.91993	21.93088	2.08434	26.02262	2.20458	21.95778	1.86021
26[a]	29.82852	23.45646	25.33982	3.12824	28.05501	1.88402	23.93009	2.11737	28.02329	2.19132	23.95845	1.87347
28[a]	32.21637	25.06513	27.34229	3.50567	30.05691	1.84988	25.92913	2.14839	30.02387	2.17997	25.95904	1.88483
30[a]	34.57967	26.72420	29.33173	3.85068	32.05885	1.81712	27.92803	2.1778	32.02440	2.17012	27.95957	1.89467
32[a]	36.92779	28.41118	31.31471	4.17611	34.06085	1.78545	29.92678	2.20581	34.02486	2.16151	29.96003	1.90329
34[a]	39.26590	30.11543	33.29399	4.48876	36.06290	1.75465	31.92540	2.23272	36.02528	2.15392	31.96045	1.91088
36[a]	41.59712	31.83115	35.27091	4.79266	38.06501	1.72456	33.92389	2.25866	38.02565	2.14716	33.96082	1.91764
38[a]	43.92346	33.55490	37.24620	5.09032	40.06719	1.69502	35.92227	2.28377	40.02599	2.14112	35.96116	1.92368
40[a]	46.24623	35.28448	39.22031	5.38339	42.06943	1.66593	37.92053	2.30816	42.02629	2.13568	37.96147	1.92912
42[a]	48.56636	37.01841	41.19352	5.67300	44.07173	1.63719	39.91868	2.33191	44.02657	2.13077	39.96175	1.93404
44[a]	50.88450	38.75567	43.16602	5.95997	46.07410	1.60874	41.91672	2.35509	46.02683	2.12629	41.96201	1.93851
46[a]	53.20112	40.49553	45.13795	6.24488	48.07654	1.58049	43.91466	2.37776	48.02706	2.12221	43.96224	1.94260
48[a]	55.51655	42.23746	47.10940	6.52815	50.07905	1.55239	45.91251	2.39997	50.02728	2.11847	45.96246	1.94634
50[a]	57.83108	43.98106	49.08046	6.81012	52.08162	1.52440	47.91026	2.42176	52.02748	2.11503	47.96266	1.94978
52[a]	60.14488	45.72601	51.05117	7.09103	54.08426	1.49646	49.90792	2.44317	54.02766	2.11185	49.96284	1.95296
54[a]	62.45811	47.47208	53.02158	7.37017	56.08697	1.46854	51.90550	2.46422	56.02783	2.10890	51.96301	1.95590
56[a]	64.77089	49.20908	54.99174	7.65040	58.08974	1.44061	53.90299	2.48494	58.02799	2.10617	53.96317	1.95864
58[a]	67.08330	50.96686	56.96167	7.92913	60.09257	1.41261	55.90040	2.50534	60.02814	2.10363	55.96332	1.96118
60[a]	69.39543	52.71530	58.93140	8.20737	62.09547	1.38453	57.89774	2.52546	62.02828	2.10125	57.96346	1.96356
62[a]	71.70732	54.46432	60.90095	8.48519	64.09842	1.34217	59.89500	2.54531	64.02841	2.09903	59.96359	1.96578
64[a]	74.01903	56.21383	62.87033	8.76265	66.10144	1.32797	61.89218	2.56490	66.02853	2.09695	61.96372	1.96786
66[a]	76.33058	57.96374	64.83957	9.03981	68.10451	1.29942	63.88930	2.58424	68.02865	2.09499	63.96383	1.96982
68[a]	78.64202	59.71402	66.80867	9.31671	70.10764	1.27066	65.88635	2.60334	70.02876	2.09315	65.96394	1.97166
70[a]	80.95335	61.46463	68.77766	9.59338	72.11082	1.24164	67.88333	2.62223	72.02886	2.09142	67.96404	1.97339
72[a]	83.26461	63.21551	70.74673	9.86987	74.11406	1.21234	69.88026	2.64090	74.02896	2.08978	69.96414	1.97503
74[a]	85.57580	64.96665	72.71531	10.14618	76.11735	1.18271	71.87712	2.65936	76.02905	2.08823	71.96423	1.97658
76[a]	87.88694	66.71800	74.68399	10.42234	78.12069	1.15271	73.87392	2.67762	78.02914	2.08676	73.96432	1.97805
78[a]	90.19805	68.46955	76.65258	10.69837	80.12408	1.12230	75.87067	2.69569	80.02922	2.08537	75.96440	1.97945
80[a]	92.50912	70.22128	78.62109	10.97429	82.12751	1.09144	77.86736	2.71358	82.02930	2.08404	77.96448	1.98077
82[a]	94.82017	71.97316	80.58954	11.25011	84.13099	1.06006	79.86401	2.73128	84.02937	2.08278	79.96456	1.98203
84[a]	97.13121	73.72518	82.55791	11.52584	86.13452	1.02812	81.86060	2.74881	86.92945	2.08158	81.96463	1.98323
86[a]	99.44223	75.47734	84.52662	11.80149	88.13808	0.99554	83.85715	2.76617	88.02951	2.08044	83.96470	1.98437
88[a]	101.75324	77.22961	86.49447	12.07707	90.14169	0.96224	85.85365	2.78337	90.02958	2.07935	85.96476	1.98546
90[a]	104.06425	78.98198	88.46267	12.35259	92.14534	0.92815	87.85010	2.80040	92.02964	2.07830	87.96483	1.98651
92[a]	106.37526	80.73445	90.43081	12.62805	94.14902	0.89315	89.84652	2.81728	94.02970	2.07731	89.96489	1.98751
94[a]	108.68626	82.48701	92.39891	12.90346	96.15274	0.85713	91.84289	2.83401	96.02976	2.07635	91.96494	1.98846
96[a]	110.99727	84.23964	94.36696	13.17882	98.15650	0.81993	93.83992	2.85058	98.02981	2.07554	93.96500	1.98938
98[a]	113.30829	85.99235	96.33497	13.45414	100.16029	0.78139	95.83552	2.87601	100.02987	2.07456	95.96505	1.99026
100[a]	115.61931	87.74513	98.30294	13.72942	102.16411	0.74128	97.83178	2.07371	102.02992	2.07371	97.96501	1.99110

[a] The complete Flügge equation yields two pair of real roots, the magnitudes of which are tabulated as p_{n1} and q_{n1}.

when n is less than about $\frac{3}{4}$ R/h. For n greater than $\frac{3}{4}$ R/h the imaginary part of one set of four roots of the simplified equation disagrees more and more with that of the complete equation (see Fig. 1a). These roots of the simplified equation remain complex, moreover, for values of n for which the corresponding roots of the complete equation are real. The calculations indicate, however, that the roots of the simplified equation may eventually become real since the imaginary part of the roots appears to decrease toward zero. The range of n considered was not large enough to verify this conjecture even for R/h as low as 10. When n is greater than R/h, the imaginary parts of the other set of four roots of the simplified and complete equations are not in agreement (see Fig. 1b). The real parts of the roots remain in relatively good agreement, however.

Table 2 Some values of the coefficients of the characteristic equation $A_0 + A_1\lambda_n^2 + A_2\lambda_n^4 + A_3\lambda_n^6 + A_4\lambda_n^8 = 0$ ($R/h = 10$, $\nu = 0.30$)

n	Coefficient	Flügge's complete eq.	Flügge's simplified eq.	Morley's eq.
20	A_0	2.54933868E10	2.54721600E10	2.54721600E10
	A_1	−2.54979341E08	−2.54817360E08	−2.55040800E08
	A_2	9.59107266E05	9.58692000E05	9.58693000E05
	A_3	−1.60276604E03	−1.59940000E03	−1.59800000E03
	A_4	1.00166458E00	1.00000000E00	1.00000000E00
100	A_0	9.80916750E15	9.80100000E15	9.80100000E15
	A_1	4.00180051E12	−3.99926003E12	−3.99940002E12
	A_2	6.00199379E08	5.99941092E08	5.99941093E08
	A_3	−4.00835870E04	−3.99994000E04	−3.99980000E04
	A_4	1.00166458E00	1.00000000E00	1.00000000E00

The roots of Morley's equations indicate similar trends except that the roots are known to always remain complex. The regions of divergence of the roots from those of the exact equation start somewhat sooner, n greater than about $\frac{3}{8}$ R/h for the first set of roots and n greater than about $\frac{1}{2}$ for the other set of roots (see Figs. 1a and 1b). These values of n also indicate the start of the regions of divergence of the roots of Morley's equation and of Flügge's approximate equation.

It is interesting to note the small differences of the coefficients of the biquartic equation which lead to such large changes in the values of the roots. Two sets of coefficients are shown in Table 2. In both cases the coefficients differ only by a fraction of a percent. The corresponding values of the roots can be found in Table 1.

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